

Convergence to the time average by stochastic regularization

O. Bernardi, F. Cardin, M. Guzzo

Dipartimento di Matematica
Università degli Studi di Padova
Via Trieste, 63 - 35121 Padova, Italy

Abstract

We compare the rate of convergence to the time average of a function over an integrable Hamiltonian flow with the one obtained by a stochastic perturbation of the same flow. Precisely, we provide detailed estimates in different Fourier norms and we prove the convergence even in a Sobolev norm for a special vanishing limit of the stochastic perturbation.

KEYWORDS: Stochastic regularization techniques, approximated first integrals, Hamiltonian Perturbation Theory, Ergodic Theory.

1 Introduction

The time averages of functions with respect to the flow of Hamiltonian systems are extensively studied in Ergodic Theory and Hamiltonian Perturbation Theory. In particular, the averages over integrable flows are commonly used as generating functions of averaging canonical transformations. In this setting it is well known since Poincaré that resonances related to the so-called small divisors represent topological obstructions to the regularity of the time averages in open sets of the phase-space. The celebrated KAM and Nekhoroshev Theorems ([6], [1], [8], [9]) overcome this problem with a refined use of algebraic as well as geometric treatment of small divisors. More recently, the so-called weak KAM theories (see [4], [7], [3]) have studied the problem by new perspectives, based on variational and PDE regularizations by viscosity techniques.

In this paper, in order to estimate the rate of convergence to the time average, we exploit a correspondence between standard viscosity regularizations of PDEs (see for example [5]) and the averages of functions over stochastic perturbations. We prove that a vanishing stochastic regularization of the time average over an integrable flow converges to the time average in a Sobolev norm accounting the first derivatives. Precisely, let us consider the integrable Hamiltonian system with Hamilton function $H(I, \varphi) := h(I)$, defined on the action-angle phase-space $A \times \mathbb{T}^n$, where $A \subseteq \mathbb{R}^n$ is open bounded and $g(I) := \nabla h(I)$ is a diffeomorphism over A such that

$$|g(I)| \leq C, \quad \max_{i,j} \left| \frac{\partial g_i}{\partial I_j}(I) \right| \leq D, \quad \left| \det \frac{\partial g}{\partial I}(I) \right| \geq m \quad (1)$$

$\forall I \in A$, for some positive constants $C, D, m > 0$. We will also denote by $\lambda > 0$ a Lipschitz constant for g in the set A .

For any smooth phase-space function $f(I, \varphi)$, we consider its finite-time average

$$G^T(I, \varphi) := \frac{1}{T} \int_0^T f(\phi^t(I, \varphi)) dt, \quad (2)$$

where $\phi^t(I, \varphi) = (I, \varphi + g(I)t)$. By denoting with

$$f(I, \varphi) := \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi}, \quad G^T(I, \varphi) := \sum_{k \in \mathbb{Z}^n} G_k^T(I) e^{ik \cdot \varphi}$$

the Fourier expansions of f and G^T , we have

$$G_k^T(I) = \begin{cases} f_k(I) & \text{if } k \cdot g(I) = 0 \\ f_k(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} & \text{if } k \cdot g(I) \neq 0 \end{cases}$$

With evidence, if $f_k \neq 0$ for a suitably large –that is generic– set of indices $k \in \mathbb{Z}^n$, the presence of small divisors $k \cdot g(I)$ represents an obstruction to the regularity both for G^T and for its limit

$$\bar{f}(I, \varphi) := \lim_{T \rightarrow +\infty} G^T(I, \varphi). \quad (3)$$

We remark that the Fourier coefficients $G_k^T(I)$ are similar to the Fourier coefficients of

$$\chi(I, \varphi) = - \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{f_k(I)}{ik \cdot g(I)} e^{ik \cdot \varphi},$$

whose ϵ -time flow ϕ_χ^ϵ formally conjugates the quasi-integrable Hamiltonian system

$$H_\epsilon(I, \varphi) = h(I) + \epsilon f(I, \varphi)$$

to its first order average

$$(H_\epsilon \circ \phi_\chi^\epsilon)(I, \varphi) = h(I) + \epsilon f_0(I) + \mathcal{O}(\epsilon^2).$$

Of course, χ and G^T are affected by the same convergence problems.

We assume from now on that f is smooth and with generic Fourier expansion. Precisely, let us introduce for any $k \in \mathbb{Z}^n$ the resonant manifold

$$\mathcal{R}_k = \{I \in A : k \cdot g(I) = 0\}, \quad (4)$$

as well as

$$\mathcal{R}_k(f) = \{I \in A : k \cdot g(I) = 0 \text{ and } |f_k(I)| > 0\}. \quad (5)$$

Then, we assume that the set

$$\mathcal{R}(f) = \bigcup_{k \in \mathbb{Z}^n \setminus 0} \mathcal{R}_k(f) \quad (6)$$

is dense in A .

We now consider the regularization of G^T based on a vanishing stochastic perturbation, previously introduced in [2] by following a technique described in [5]. More precisely, let (Ω, \mathcal{F}, P) be a probability

space and $w_t : \Omega \rightarrow \mathbb{R}^n$ a n -dimensional Wiener process. Then, we obtain a stochastic differential equation by perturbing the Hamilton equations with a white noise

$$\begin{cases} \dot{I}_t = 0 \\ \dot{\varphi}_t = g(I) + 2\nu\dot{w}_t \end{cases} \quad (7)$$

whose flow is $\Phi_\nu^t(I, \varphi, \omega) = (I, \varphi + g(I)t + 2\nu w_t(\omega))$. As in [2], for $\mu, \nu > 0$ we introduce

$$F^{\mu, \nu}(I, \varphi) := \mu M_{(I, \varphi)} \left(\int_0^{+\infty} f(\Phi_\nu^t(I, \varphi, \omega)) e^{-\mu t} dt \right). \quad (8)$$

In the previous formula, $M_{(I, \varphi)}$ represents, for (I, φ) fixed, the average on all the trajectories of the Brownian motion (7), while the exponential damping $e^{-\mu t}$ allows us to interpret $F^{\mu, \nu}$ as an effective average over a time interval of some multiples of $1/\mu$ (see [2]). Moreover, in this paper this factor will play an essential role to ensure the convergence for $(\mu, \nu) \rightarrow (0, 0)$.

In order to study the convergence properties of G^T and $F^{\mu, \nu}$ to the time average \bar{f} , we introduce specific norms on $A \times \mathbb{T}^n$. In more detail, for any function $u(I, \varphi) = \sum_{k \in \mathbb{Z}^n} u_k(I) e^{ik \cdot \varphi}$ on $A \times \mathbb{T}^n$, the uniform Fourier norm

$$|u|^\infty := \sum_{k \in \mathbb{Z}^n} \sup_{I \in A} |u_k(I)| \quad (9)$$

as well as the norms obtained with averages over the action space

$$|u|^0 := \sum_{k \in \mathbb{Z}^n} \int_A |u_k(I)| dI \quad (10)$$

and

$$|u|^1 := |u|^0 + \sum_{k \in \mathbb{Z}^n} \sum_{j=1}^n \int_A \left(\left| \frac{\partial u_k}{\partial I_j}(I) \right| + |k_j u_k(I)| \right) dI. \quad (11)$$

Let us remark that, by considering the usual L^1 and Sobolev $W^{1,1}$ norms on $A \times \mathbb{T}^n$, in particular

$$\|u\|_{W^{1,1}} = \|u\|_{L^1} + \sum_{j=1}^n \left(\left\| \frac{\partial u}{\partial I_j} \right\|_{L^1} + \left\| \frac{\partial u}{\partial \varphi_j} \right\|_{L^1} \right)$$

and we have

$$\frac{1}{(2\pi)^n} \|u\|_{W^{1,1}} \leq |u|^1 \leq \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \|u_k(I) e^{ik \cdot \varphi}\|_{W^{1,1}}$$

The paper presents the following results. In Proposition 2.1, we first prove that for a generic f , both G^T and $F^{\mu, \nu}$ do not converge to \bar{f} in the uniform norm $|\cdot|^\infty$, but they converge to \bar{f} in the $|\cdot|^0$ norm. The main result, given in Proposition 2.2, concerns with the stronger norm $|\cdot|^1$: while the finite time average G^T does not converge to \bar{f} in the $|\cdot|^1$ norm, we have

$$\lim_{i \rightarrow +\infty} |F^{\mu_i, \nu_i} - \bar{f}|^1 = 0$$

for any sequence μ_i, ν_i converging to zero and such that $\lim_{i \rightarrow +\infty} \frac{\mu_i}{\nu_i} = 0$.

The paper is organized as follows. In Section 2 we state Propositions 2.1 and 2.2 on the convergence of the regularized averages. Section 3 is devoted to proofs.

2 Convergence results

Let us consider a phase-space function $f(I, \varphi)$, its finite time average G^T and its time average \bar{f} defined in (2) and (3) respectively. In [2] we have introduced two different approximations of G^T offering a better notion of approximated first integral. The first one is

$$F^\mu(I, \varphi) := \mu \int_0^{+\infty} f(\phi^t(I, \varphi)) e^{-\mu t} dt, \quad (12)$$

with $\mu = 1/T$, while the second one is $F^{\mu, \nu}$, whose definition recalled in Section 1 (see (8)), comes from the above stochastic setting. Let us remark that F^μ represents an intermediate step between G^T and $F^{\mu, \nu}$, in the sense that it is an exponentially damped average of f with respect to the integrable flow, that is, $F^\mu = F^{\mu, 0}$. However, the improvement in the convergence properties to \bar{f} is obtained only for $\nu > 0$ (see the propositions below).

We first discuss the convergence of the above approximated first integrals G^T , F^μ and $F^{\mu, \nu}$ to the time average \bar{f} both in the uniform Fourier norm $|\cdot|^\infty$ —see (9)—and in the action-averages based norm $|\cdot|^0$ given in (10). In particular, we prove the next

Proposition 2.1 *The functions G^T , F^μ and $F^{\mu, \nu}$ do not uniformly Fourier converge to \bar{f} in any set $B \times \mathbb{T}^n$ with $B \subseteq A$ open, but converge to \bar{f} in the $|\cdot|^0$ norm on $A \times \mathbb{T}^n$. Precisely, we have*

$$|G^T - \bar{f}|^0 \leq \frac{4C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{3 + \log(\|k\|TC)}{T} \quad (13)$$

$$|F^\mu - \bar{f}|^0 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\|k\|} \left(1 + \log \frac{\|k\|C}{\mu} \right) \quad (14)$$

$$|F^{\mu, \nu} - \bar{f}|^0 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\|k\|} \left(1 + \log \frac{\|k\|C}{\mu + \nu \|k\|^2} \right). \quad (15)$$

As it arises from the previous proposition, the three different finite-time approximations G^T , F^μ and $F^{\mu, \nu}$ behave in the same way with respect to the $|\cdot|^\infty$ and $|\cdot|^0$ norms. Indeed, the difference consists in the convergence in the $|\cdot|^1$ norm given in (11). In such a case, the G^T , F^μ do not converge to \bar{f} , and it is remarkable that the convergence of $F^{\mu, \nu}$ is obtained only in a special limit of vanishing stochastic perturbation, as stated in the proposition below.

Proposition 2.2 *The functions G^T and F^μ do not converge to \bar{f} in the $|\cdot|^1$ norm on any set $B \times \mathbb{T}^n$ with $B \subseteq A$ open. Differently, for any $\mu, \nu > 0$ the function $F^{\mu, \nu}$ satisfies*

$$|F^{\mu, \nu} - \bar{f}|^1 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\mu \left[1 + \log \frac{\|k\|C}{\mu + \nu \|k\|^2} \right] \left((1+n) |f_k|^\infty + \sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) + \frac{1}{2} n^2 \pi D \frac{\mu}{\mu + \nu \|k\|^2} |f_k|^\infty \right) \quad (16)$$

on $A \times \mathbb{T}^n$. In particular, for any sequence $\mu_i, \nu_i > 0$ converging to zero and such that

$$\lim_{i \rightarrow +\infty} \frac{\mu_i}{\nu_i} = 0,$$

we have

$$\lim_{i \rightarrow +\infty} |F^{\mu_i, \nu_i} - \bar{f}|^1 = 0.$$

Let us remark that the convergence of $F^{\mu, \nu}$ to \bar{f} requires a restriction of the sub-sequences μ_i, ν_i because in (16) we find contributions proportional to $\frac{\mu}{\mu + \nu \|k\|^2}$, while the contributions $\mu \log \frac{\|k\|C}{\mu + \nu \|k\|^2}$ which are dominant in (15) converge for $(\mu, \nu) \rightarrow (0, 0)$.

The proofs of Propositions 2.1, 2.2 are reported in Section 3.

3 Proofs

The different time averages (2), (12) and (8) can be alternatively expressed in terms of their Fourier coefficients, as discussed in the following technical

Lemma 3.1 *Let us consider*

$$f(I, \varphi) = \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi}. \quad (17)$$

The Fourier coefficients of

$$G^T(I, \varphi) = \sum_{k \in \mathbb{Z}^n} G_k^T(I) e^{ik \cdot \varphi}, \quad F^\mu(I, \varphi) = \sum_{k \in \mathbb{Z}^n} F_k^\mu(I) e^{ik \cdot \varphi}, \quad F^{\mu, \nu}(I, \varphi) = \sum_{k \in \mathbb{Z}^n} F_k^{\mu, \nu}(I) e^{ik \cdot \varphi}$$

are respectively

$$G_k^T(I) = \begin{cases} f_k(I) & \text{if } k \cdot g(I) = 0 \\ f_k(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} & \text{if } k \cdot g(I) \neq 0 \end{cases} \quad (18)$$

$$F_k^\mu(I) = -\mu \frac{f_k(I)}{ik \cdot g(I) - \mu} \quad (19)$$

and

$$F_k^{\mu, \nu}(I) = -\mu \frac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} \quad (20)$$

Proof. The first equality easily follows from (2) and (17) by direct calculations. Indeed

$$\begin{aligned} G^T(I, \phi) &= \frac{1}{T} \int_0^T f(\phi^t(I, \varphi)) dt = \frac{1}{T} \int_0^T \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} e^{ik \cdot g(I)t} dt \\ &= \frac{1}{T} \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} \int_0^T e^{ik \cdot g(I)t} dt. \end{aligned}$$

Moreover, from

$$\frac{1}{T} \int_0^T e^{ik \cdot g(I)t} dt = \begin{cases} 1 & \text{if } k \cdot g(I) = 0 \\ \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} & \text{if } k \cdot g(I) \neq 0 \end{cases}$$

we immediately obtain formula (18). Similarly for (19)

$$\begin{aligned} F^\mu(I, \varphi) &= \mu \int_0^{+\infty} f(\phi^t(I, \varphi)) e^{-\mu t} dt = \mu \int_0^{+\infty} \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} e^{ik \cdot g(I)t - \mu t} dt \\ &= \mu \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} \int_0^{+\infty} e^{(ik \cdot g(I) - \mu)t} dt = -\mu \sum_{k \in \mathbb{Z}^n} \frac{f_k(I)}{ik \cdot g(I) - \mu} e^{ik \cdot \varphi}. \end{aligned}$$

We conclude by proving the equality (20). We first take into account (8), so that

$$\int_0^{+\infty} f(\Phi_\nu^t(I, \varphi, \omega)) e^{-\mu t} dt = \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} \int_0^{+\infty} e^{(ik \cdot g(I) - \mu)t} e^{2i\nu k \cdot w_t(\omega)} dt.$$

As a consequence –see (8)– we obtain

$$\begin{aligned}
F^{\mu,\nu}(I, \varphi) &= \mu M_{(I, \varphi)} \left(\int_0^{+\infty} f(\Phi_\nu^t(I, \varphi, \omega)) e^{-\mu t} dt \right) \\
&= \mu \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} \int_\Omega \left[\int_0^{+\infty} e^{(ik \cdot g(I) - \mu)t} e^{2i\nu k \cdot w_t(\omega)} dt \right] P(d\omega) \\
&= \mu \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} \int_0^{+\infty} \left[e^{(ik \cdot g(I) - \mu)t} \int_\Omega e^{2i\nu k \cdot w_t(\omega)} P(d\omega) \right] dt.
\end{aligned} \tag{21}$$

Since $w_t : \Omega \rightarrow \mathbb{R}^n$ is a n -dimensional Wiener process, the corresponding covariance matrix $R(t) = R_{ij}(t) = t\delta_{ij}$ and therefore

$$\int_\Omega e^{2i\nu k \cdot w_t(\omega)} P(d\omega) = e^{-\nu \|k\|^2 t}.$$

Therefore, from equation (21) we have

$$F^{\mu,\nu}(I, \varphi) = \mu \sum_{k \in \mathbb{Z}^n} f_k(I) e^{ik \cdot \varphi} \int_0^{+\infty} e^{(ik \cdot g(I) - \mu - \nu \|k\|^2)t} dt = -\mu \sum_{k \in \mathbb{Z}^n} \frac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} e^{ik \cdot \varphi}.$$

□

The next sections are devoted to the convergence results, in three different norms, of G^T , F^μ and $F^{\mu,\nu}$ to the time average \bar{f} defined in (3). From (18), we immediately obtain $\bar{f} = \sum_{k \in \mathbb{Z}^n} \bar{f}_k(I) e^{ik \cdot \varphi}$, with

$$\bar{f}_k(I) = \begin{cases} f_k(I) & \text{if } k \cdot g(I) = 0 \\ 0 & \text{if } k \cdot g(I) \neq 0 \end{cases} \tag{22}$$

3.1 Proof of Proposition 2.1

We start by proving that G^T does not converge to \bar{f} in the uniform Fourier norm. Let us consider

$$(G^T - \bar{f})(I, \varphi) := \sum_{k \in \mathbb{Z}^n} (G^T - \bar{f})_k(I) e^{ik \cdot \varphi}.$$

From (18) and (22) we immediately obtain

$$(G^T - \bar{f})_k(I) = \begin{cases} 0 & \text{if } k \cdot g(I) = 0 \\ f_k(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} & \text{if } k \cdot g(I) \neq 0 \end{cases} \tag{23}$$

Since the set $\mathcal{R}(f)$ defined in (6) is dense, there exists a dense set of points $\bar{I} \in A$ such that $\bar{k} \cdot g(\bar{I}) = 0$ and $|f_{\bar{k}}(\bar{I})| > 0$ for some $\bar{k} \in \mathbb{Z}^n \setminus 0$. Since g is a diffeomorphism, we have

$$\lim_{I \notin \mathcal{R}_{\bar{k}}, I \rightarrow \bar{I}} \left| \frac{e^{i\bar{k} \cdot g(I)T} - 1}{i\bar{k} \cdot g(I)T} \right| = \lim_{J \rightarrow 0} \left| \frac{e^{iJT} - 1}{iJT} \right| = \lim_{J \rightarrow 0} \frac{\sqrt{2[1 - \cos(JT)]}}{|JT|} = 1,$$

and also

$$\sup_{I \in A \setminus \mathcal{R}_{\bar{k}}} \left| f_{\bar{k}}(I) \frac{e^{i\bar{k} \cdot g(I)T} - 1}{i\bar{k} \cdot g(I)T} \right| \geq \lim_{I \notin \mathcal{R}_{\bar{k}}, I \rightarrow \bar{I}} \left| f_{\bar{k}}(I) \frac{e^{i\bar{k} \cdot g(I)T} - 1}{i\bar{k} \cdot g(I)T} \right| = |f_{\bar{k}}(\bar{I})|.$$

As a consequence,

$$|G^T - \bar{f}|^\infty = \sum_{k \in \mathbb{Z}^n} \sup_{I \in A} |(G^T - \bar{f})_k(I)| \geq |f_{\bar{k}}(\bar{I})| > 0$$

that is, G^T does not uniformly Fourier converge to \bar{f} in any set $B \times \mathbb{T}^n$ with $B \subseteq A$ open.

We proceed with the same discussion for F^μ . By denoting

$$(F^\mu - \bar{f})(I, \varphi) := \sum_{k \in \mathbb{Z}^n} (F^\mu - \bar{f})_k(I) e^{ik \cdot \varphi},$$

from (19) and (22) we have

$$(F^\mu - \bar{f})_k(I) = \begin{cases} 0 & \text{if } k \cdot g(I) = 0 \\ -\mu \frac{f_k(I)}{ik \cdot g(I) - \mu} & \text{if } k \cdot g(I) \neq 0 \end{cases} \quad (24)$$

By considering as before $\bar{I} \in \mathcal{R}(f)$ such that $\bar{k} \cdot g(\bar{I}) = 0$ and $|f_{\bar{k}}(\bar{I})| > 0$, for some $\bar{k} \neq 0$, from

$$\lim_{I \notin \mathcal{R}_{\bar{k}}, I \rightarrow \bar{I}} \left| -\mu \frac{f_{\bar{k}}(I)}{ik \cdot g(I) - \mu} \right| = |f_{\bar{k}}(\bar{I})|$$

we have

$$|F^\mu - \bar{f}|^\infty = \sum_{k \in \mathbb{Z}^n} \sup_{I \in A} |(F^\mu - \bar{f})_k(I)| \geq |f_{\bar{k}}(\bar{I})| > 0$$

that is, F^μ does not uniformly Fourier converge to \bar{f} in any set $B \times \mathbb{T}^n$ with $B \subseteq A$ open.

We conclude the first part of the proof by showing that also $F^{\mu, \nu}$ does not uniformly Fourier converge to \bar{f} . Indeed, in such a case, formulas (20) and (22) give

$$(F^{\mu, \nu} - \bar{f})_k(I) = \begin{cases} f_k(I) \left[\frac{\mu}{\mu + \nu \|k\|^2} - 1 \right] & \text{if } k \cdot g(I) = 0 \\ -\mu \frac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} & \text{if } k \cdot g(I) \neq 0 \end{cases} \quad (25)$$

By considering again $\bar{k} \cdot g(\bar{I}) = 0$ and $|f_{\bar{k}}(\bar{I})| > 0$ with $\bar{k} \neq 0$, and sequences $(\mu_i, \nu_i) \rightarrow 0$, we discuss the following two cases.

(i) If $\lim_{i \rightarrow +\infty} \frac{\nu_i}{\mu_i} = 0$, we have

$$\lim_{i \rightarrow +\infty} \lim_{I \notin \mathcal{R}_{\bar{k}}, I \rightarrow \bar{I}} |(F^{\mu_i, \nu_i} - \bar{f})_{\bar{k}}(I)| = \lim_{i \rightarrow +\infty} \left| \mu_i \frac{f_{\bar{k}}(\bar{I})}{\mu_i + \nu_i \|\bar{k}\|^2} \right| = |f_{\bar{k}}(\bar{I})|.$$

(ii) On the contrary, if the sequence $\frac{\nu_i}{\mu_i}$ does not converge to zero, we consider

$$|(F^{\mu_i, \nu_i} - \bar{f})_{\bar{k}}(\bar{I})| = |f_{\bar{k}}(\bar{I})| \left| \left[\frac{\mu_i}{\mu_i + \nu_i \|\bar{k}\|^2} - 1 \right] \right| = |f_{\bar{k}}(\bar{I})| \frac{\nu_i \|\bar{k}\|^2}{\mu_i + \nu_i \|\bar{k}\|^2}$$

which does not converge to zero as i tends to infinity.

As a consequence of all previous cases, we conclude that $F^{\mu, \nu}$ does not uniformly Fourier converge to \bar{f} in any set $B \times \mathbb{T}^n$ with $B \subseteq A$ open.

We proceed by discussing the convergence to \bar{f} in the $|\cdot|^0$ norm. Since $g : A \rightarrow \mathbb{R}^n$ is a diffeomorphism, the set of all resonances $\mathcal{R} := \bigcup_{k \in \mathbb{Z}^n \setminus 0} \mathcal{R}_k$ has measure zero. Consequently, the norm $|\cdot|^0$ can be rewritten as

$$|u|^0 = \sum_{k \in \mathbb{Z}^n} \int_{\bar{A}} |u_k(I)| dI$$

where

$$\tilde{A} := A \setminus \mathcal{R} = \{I \in A : k \cdot g(I) \neq 0 \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}\}. \quad (26)$$

We first prove $\lim_{T \rightarrow +\infty} |G^T - \bar{f}|^0 = 0$. From (18) and (22) we immediately obtain

$$(G^T - \bar{f})_k(I) = f_k(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} \quad \forall I \in \tilde{A}$$

so that

$$\begin{aligned} \int_{\tilde{A}} |(G^T - \bar{f})_k(I)| dI &= \int_{\tilde{A}} \left| f_k(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} \right| dI \\ &\leq |f_k|^\infty \int_{\tilde{A}} \frac{\sqrt{\sin^2(k \cdot g(I)T) + [\cos(k \cdot g(I)T) - 1]^2}}{|k \cdot g(I)|T} dI = |f_k|^\infty \int_{\tilde{A}} \frac{\sqrt{2[1 - \cos(k \cdot g(I)T)]}}{|k \cdot g(I)|T} dI. \end{aligned}$$

Using the change of variables

$$I \mapsto J := g(I) \quad (27)$$

and assumption (1), we obtain

$$\int_{\tilde{A}} |(G^T - \bar{f})_k(I)| dI \leq |f_k|^\infty \int_{\tilde{A}} \frac{\sqrt{2[1 - \cos(k \cdot g(I)T)]}}{|k \cdot g(I)|T} dI \leq \frac{|f_k|^\infty}{m} \int_{g(\tilde{A})} \frac{\sqrt{2[1 - \cos(k \cdot JT)]}}{|k \cdot J|T} dJ. \quad (28)$$

Let now $\tilde{e}_1, \dots, \tilde{e}_n$ be an orthonormal basis of \mathbb{R}^n with $k \in \langle \tilde{e}_2, \dots, \tilde{e}_n \rangle^\perp$ and R a rotation matrix such that $Rk = \|k\| \tilde{e}_1$ (the dependence of the basis and the rotation matrix on $k \in \mathbb{Z}^n$ is here omitted). By the further change of variables

$$J \mapsto x := RJ \quad (29)$$

the quantity $k \cdot J$ in (28) becomes $k \cdot J = \|k\| x_1$, and for any x in the integration domain $Rg(\tilde{A})$ we have $x = Rg(I)$ with $I \in \tilde{A}$ and $\|x\| \leq \|g(I)\| \leq C$. As a consequence, we obtain

$$\begin{aligned} \int_{\tilde{A}} |(G^T - \bar{f})_k(I)| dI &\leq \frac{|f_k|^\infty}{m} \int_{g(\tilde{A})} \frac{\sqrt{2[1 - \cos(k \cdot JT)]}}{|k \cdot J|T} dJ = \frac{|f_k|^\infty}{m} \int_{Rg(\tilde{A})} \frac{\sqrt{2[1 - \cos(\|k\| x_1 T)]}}{\|k\| |x_1| T} dx_1 \dots dx_n \\ &\leq \frac{|f_k|^\infty C^{n-1}}{m} \int_{-C}^C \frac{\sqrt{2[1 - \cos(\|k\| x_1 T)]}}{\|k\| |x_1| T} dx_1 = \frac{|f_k|^\infty C^{n-1}}{m \|k\| T} \int_{-\|k\| CT}^{\|k\| CT} \frac{\sqrt{2(1 - \cos y)}}{|y|} dy \\ &= \frac{2|f_k|^\infty C^{n-1}}{m \|k\| T} \int_{-\|k\| CT/2}^{\|k\| CT/2} \left| \frac{\sin y}{y} \right| dy = \frac{4|f_k|^\infty C^{n-1}}{m \|k\| T} \int_0^{\|k\| CT/2} \left| \frac{\sin y}{y} \right| dy \\ &\leq \frac{4|f_k|^\infty C^{n-1}}{m \|k\| T} \int_0^{2\pi} \left| \frac{\sin y}{y} \right| dy + \frac{4|f_k|^\infty C^{n-1}}{m \|k\| T} \int_{2\pi}^{\|k\| TC/2} \frac{1}{y} dy \leq \frac{4|f_k|^\infty C^{n-1}}{m \|k\| T} [l_0 + \log(\|k\| TC)] \quad (30) \end{aligned}$$

with $l_0 := \int_0^{2\pi} \left| \frac{\sin y}{y} \right| dy \leq 3$. Consequently,

$$|G^T - \bar{f}|^0 \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{4|f_k|^\infty C^{n-1}}{m \|k\| T} [3 + \log(\|k\| TC)]$$

proving that G^T converges to \bar{f} in the $|\cdot|^0$ norm.

We conclude the proof with the convergence of F^μ and $F^{\mu, \nu}$ to \bar{f} . By using formulas (20) and (22), we have

$$(F^{\mu, \nu} - \bar{f})_k(I) = -\mu \frac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} \quad \forall I \in \tilde{A}.$$

Hence

$$\begin{aligned} \int_{\tilde{A}} |(F^{\mu,\nu} - \bar{f})_k(I)| dI &= \mu \int_{\tilde{A}} \frac{|f_k(I)|}{\sqrt{(\mu + \nu \|k\|^2)^2 + (k \cdot g(I))^2}} dI \\ &\leq \mu |f_k|^\infty \int_{\tilde{A}} \frac{1}{\sqrt{(\mu + \nu \|k\|^2)^2 + (k \cdot g(I))^2}} dI \end{aligned}$$

The same changes of variables of the previous case, see (27) and (29), provide

$$\begin{aligned} \int_{\tilde{A}} |(F^{\mu,\nu} - \bar{f})_k(I)| dI &\leq \frac{\mu |f_k|^\infty C^{n-1}}{m} \int_{-C}^C \frac{1}{\sqrt{(\mu + \nu \|k\|^2)^2 + \|k\|^2 x_1^2}} dx_1 \\ &= \frac{\mu |f_k|^\infty C^{n-1}}{m \|k\|} \int_{-\frac{\|k\|C}{\mu + \nu \|k\|^2}}^{\frac{\|k\|C}{\mu + \nu \|k\|^2}} \frac{1}{\sqrt{1+x^2}} dx = \frac{2\mu |f_k|^\infty C^{n-1}}{m \|k\|} \int_0^{\frac{\|k\|C}{\mu + \nu \|k\|^2}} \frac{1}{\sqrt{1+x^2}} dx \\ &= \frac{2\mu |f_k|^\infty C^{n-1}}{m \|k\|} \left[\int_0^1 \frac{1}{\sqrt{1+x^2}} dx + \int_1^{\frac{\|k\|C}{\mu + \nu \|k\|^2}} \frac{1}{\sqrt{1+x^2}} dx \right] \leq \frac{2\mu |f_k|^\infty C^{n-1}}{m \|k\|} \left[l_1 + \int_1^{\frac{\|k\|C}{\mu + \nu \|k\|^2}} \frac{1}{x} dx \right] \\ &= \frac{2\mu |f_k|^\infty C^{n-1}}{m \|k\|} \left[l_1 + \log \frac{\|k\|C}{\mu + \nu \|k\|^2} \right] \end{aligned} \quad (31)$$

with $l_1 := \operatorname{arcsinh} 1 \leq 1$. Consequently

$$|F^{\mu,\nu} - \bar{f}|^0 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\|k\|} \left[1 + \log \frac{\|k\|C}{\mu + \nu \|k\|^2} \right]$$

and for $\nu = 0$ we obtain also (14).

Inequalities (15) and (14) respectively prove that $F^{\mu,\nu}$ converges to \bar{f} for $(\mu, \nu) \rightarrow (0, 0)$ and F^μ converges to \bar{f} for $\mu \rightarrow 0$ in the $|\cdot|^0$ norm. \square

3.2 Proof of Proposition 2.2

Let us consider any open set $B \subseteq A$. Since $g : A \rightarrow \mathbb{R}^n$ is a diffeomorphism, the $|\cdot|^1$ norm in $B \times \mathbb{T}^n$ –see (11)– can be rewritten as

$$|u|^1 = \sum_{k \in \mathbb{Z}^n} \left\{ \int_{\tilde{B}} |u_k(I)| dI + \sum_{j=1}^n \int_{\tilde{B}} \left(\left| \frac{\partial u_k}{\partial I_j}(I) \right| + |k_j u_k(I)| \right) dI \right\}$$

with $\tilde{B} = B \cap \tilde{A}$, see (26).

We first prove that G^T does not converge to \bar{f} in the set $B \times \mathbb{T}^n$. It is sufficient to prove that there exists $\epsilon > 0$ such that for any large T we have

$$\sum_{j=1}^n \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} \left| \left(\frac{\partial G_k^T}{\partial I_j} - \frac{\partial \bar{f}_k}{\partial I_j} \right) (I) \right| dI > \epsilon. \quad (32)$$

From (18) and (22), for any $I \in \tilde{B}$ we have

$$(G^T - \bar{f})_k(I) = \begin{cases} f_k(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases}$$

so that

$$\left(\frac{\partial G_k^T}{\partial I_j} - \frac{\partial \bar{f}_k}{\partial I_j} \right) (I) = \begin{cases} \frac{\partial f_k}{\partial I_j}(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} + f_k(I) \frac{\partial}{\partial I_j} \left(\frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} \right) & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases} \quad (33)$$

We notice that the first addendum in (33) tends to 0, that is

$$\lim_{T \rightarrow +\infty} \int_{\tilde{B}} \left| \frac{\partial f_k}{\partial I_j}(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} \right| dI = 0.$$

Indeed, by using the changes of variables (27) and (29) as in the proof of Proposition 2.1 –see also (30)– we obtain

$$\int_{\tilde{B}} \left| \frac{\partial f_k}{\partial I_j}(I) \frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} \right| dI \leq \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \frac{4C^{n-1}}{m\|k\|T} [l_0 + \log(\|k\|TC)].$$

As a consequence, it remains to study the other term of the equality (33), precisely

$$\begin{aligned} & \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} |f_k(I)| \left| \frac{\partial}{\partial I_j} \left(\frac{e^{ik \cdot g(I)T} - 1}{ik \cdot g(I)T} \right) \right| dI \\ &= \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} |f_k(I)| \left| \frac{\partial}{\partial I_j} (ik \cdot g(I)) \right| \frac{\sqrt{2 + (k \cdot g)^2 T^2 - 2k \cdot gT \sin(k \cdot gT) - 2 \cos(k \cdot gT)}}{(k \cdot g)^2 T} dI \\ &= \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} |f_k(I)| \left\| \frac{\partial g^T}{\partial I} k \right\|_1 \frac{\sqrt{2 + (k \cdot g)^2 T^2 - 2k \cdot gT \sin(k \cdot gT) - 2 \cos(k \cdot gT)}}{(k \cdot g)^2 T} dI \end{aligned}$$

where

$$\left\| \frac{\partial g^T}{\partial I} k \right\|_1 := \sum_{j=1}^n \left| \left(\frac{\partial g^T}{\partial I} k \right)_j \right| = \sum_{j=1}^n \left| \sum_{i=1}^n \frac{\partial g_i}{\partial I_j} k_i \right| = \sum_{j=1}^n \left| \frac{\partial}{\partial I_j} (ik \cdot g(I)) \right|.$$

Since $\mathcal{R}(f)$ is dense in A , there exists $\bar{I} \in B \cap \mathcal{R}(f)$ such that $k \cdot g(\bar{I}) = 0$ and $|f_k(\bar{I})| > 0$ for some $k \in \mathbb{Z}^n \setminus 0$. In particular, there exist $\delta, \lambda_1, \lambda_2 > 0$ (independent of T) such that the closed ball

$$B_\delta(\bar{I}) = \{I : \|I - \bar{I}\| \leq \delta\}$$

is contained in B , and also for any $I \in B_\delta(\bar{I})$ we have

$$|f_k(I)| \geq \lambda_1$$

and

$$\min_{\|u\|=1} \left\| \frac{\partial g^T}{\partial I} u \right\|_1 \geq \lambda_2.$$

Let us remark that the constant λ_1 satisfies $0 < \lambda_1 \leq |f_k|^\infty$. The constant λ_2 is indeed strictly positive, since otherwise there would exist $u \neq 0$ with $\frac{\partial g^T}{\partial I} u = 0$, which is in contradiction with (1). From (1), there exists also a constant $M > 0$ such that

$$\left| \det \frac{\partial g}{\partial I}(I) \right| \leq M \quad (34)$$

for any $I \in A$. As a consequence, we have

$$\begin{aligned} & \sum_{j=1}^n \sum_{\tilde{k} \in \mathbb{Z}^n} \int_{\tilde{B}} |f_{\tilde{k}}(I)| \left| \frac{\partial}{\partial I_j} \left(\frac{e^{i\tilde{k} \cdot g(I)T} - 1}{i\tilde{k} \cdot g(I)T} \right) \right| dI \\ & \geq \lambda_1 \lambda_2 \|k\| \int_{B_\delta(\bar{I})} \frac{\sqrt{2 + (k \cdot g)^2 T^2 - 2k \cdot gT \sin(k \cdot gT) - 2 \cos(k \cdot gT)}}{(k \cdot g)^2 T} dI. \end{aligned}$$

By performing the change of variables $J := g(I)$ and using (34), the above term has the lower bound

$$\frac{\lambda_1 \lambda_2}{M} \|k\| \int_{g(B_\delta(\bar{I}))} \frac{\sqrt{2 + (k \cdot J)^2 T^2 - 2k \cdot JT \sin(k \cdot JT) - 2 \cos(k \cdot JT)}}{(k \cdot J)^2 T} dJ,$$

which, using the additional change of variables $x := RJ$ as in (29), equals to

$$\frac{\lambda_1 \lambda_2}{M} \|k\| \int_{Rg(B_\delta(\bar{I}))} \frac{\sqrt{2 + \|k\|^2 x_1^2 T^2 - 2\|k\| x_1 T \sin(\|k\| x_1 T) - 2 \cos(\|k\| x_1 T)}}{\|k\|^2 x_1^2 T} dx.$$

We consider $\tilde{\delta} > 0$ possibly depending on k, \bar{I} (but independent of T) such that

$$\left\{ x : \max_{j=1, \dots, n} |x_j - Rg(\bar{I})_j| \leq \tilde{\delta} \right\} \subseteq Rg(B_\delta(\bar{I})),$$

so that we have

$$\begin{aligned} & \frac{\lambda_1 \lambda_2}{M} \|k\| \int_{Rg(B_\delta(\bar{I}))} \frac{\sqrt{2 + \|k\|^2 x_1^2 T^2 - 2\|k\| x_1 T \sin(\|k\| x_1 T) - 2 \cos(\|k\| x_1 T)}}{\|k\|^2 x_1^2 T} dx \\ & \geq \frac{\lambda_1 \lambda_2}{M} \|k\| \tilde{\delta}^{n-1} \int_{Rg(\bar{I})_1 - \tilde{\delta}}^{Rg(\bar{I})_1 + \tilde{\delta}} \frac{\sqrt{2 + \|k\|^2 x_1^2 T^2 - 2\|k\| x_1 T \sin(\|k\| x_1 T) - 2 \cos(\|k\| x_1 T)}}{\|k\|^2 x_1^2 T} dx_1 \\ & = \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{\|k\|T(Rg(\bar{I})_1 - \tilde{\delta})}^{\|k\|T(Rg(\bar{I})_1 + \tilde{\delta})} \frac{\sqrt{2 + y^2 - 2y \sin y - 2 \cos y}}{y^2} dy. \end{aligned}$$

We remark that, since the change of variables (29) is performed by a matrix R such that $Rk = \|k\| \tilde{e}_1$, so that

$$Rg(\bar{I})_1 = \tilde{e}_1 \cdot Rg(\bar{I}) = \frac{1}{\|k\|} Rk \cdot Rg(\bar{I}) = \frac{1}{\|k\|} k \cdot g(\bar{I}) = 0,$$

we have

$$\frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{\|k\|T(Rg(\bar{I})_1 - \tilde{\delta})}^{\|k\|T(Rg(\bar{I})_1 + \tilde{\delta})} \frac{\sqrt{2 + y^2 - 2y \sin y - 2 \cos y}}{y^2} dy = \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{-\|k\|T\tilde{\delta}}^{\|k\|T\tilde{\delta}} \frac{\sqrt{2 + y^2 - 2y \sin y - 2 \cos y}}{y^2} dy.$$

Since for any $y \in \mathbb{R}$ we have

$$2 + y^2 - 2y \sin y - 2 \cos y \geq \frac{y^4}{4(1 + y^2)},$$

we conclude

$$\frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{-\|k\|T\tilde{\delta}}^{\|k\|T\tilde{\delta}} \frac{\sqrt{2 + y^2 - 2y \sin y - 2 \cos y}}{y^2} dy \geq \frac{\lambda_1 \lambda_2}{2M} \tilde{\delta}^{n-1} \int_{-\|k\|T\tilde{\delta}}^{\|k\|T\tilde{\delta}} \frac{1}{\sqrt{1 + y^2}} dy$$

$$= \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_0^{\|k\| T \tilde{\delta}} \frac{1}{\sqrt{1+y^2}} dy = \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \operatorname{arcsinh}(\|k\| T \tilde{\delta}).$$

Since

$$\lim_{T \rightarrow +\infty} \operatorname{arcsinh}(\|k\| T \tilde{\delta}) = +\infty,$$

with a suitable definition of ϵ , one immediately obtains (32).

We proceed by proving that F^μ does not converge to \bar{f} in the set $B \times \mathbb{T}^n$. It is sufficient to prove that there exists $\epsilon > 0$ such that for any small μ we have

$$\sum_{j=1}^n \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} \left| \left(\frac{\partial F_k^\mu}{\partial I_j} - \frac{\partial \bar{f}_k}{\partial I_j} \right) (I) \right| dI > \epsilon. \quad (35)$$

From (24), for any $I \in \tilde{B}$ we have

$$(F^\mu - \bar{f})_k(I) = \begin{cases} F_k^\mu(I) = -\frac{\mu f_k(I)}{ik \cdot g(I) - \mu} & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases}$$

so that we have to estimate

$$\sum_{j=1}^n \sum_{k \in \mathbb{Z}^n \setminus 0} \int_{\tilde{B}} \left| \frac{\partial F_k^\mu}{\partial I_j}(I) \right| dI.$$

By direct computations we obtain

$$\begin{aligned} \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n \setminus 0} \int_{\tilde{B}} \left| \frac{\partial F_k^\mu}{\partial I_j}(I) \right| dI &= \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n \setminus 0} \mu \int_{\tilde{B}} \frac{1}{|ik \cdot g(I) - \mu|^2} \left| \frac{\partial f_k}{\partial I_j}(I) (ik \cdot g(I) - \mu) - f_k(I) \frac{\partial}{\partial I_j} (ik \cdot g(I)) \right| dI \\ &= \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n \setminus 0} \mu \int_{\tilde{B}} \frac{1}{|ik \cdot g(I) - \mu|^2} \sqrt{\mu^2 \left(\frac{\partial f_k}{\partial I_j} \right)^2 + \left(k \cdot g(I) \frac{\partial f_k}{\partial I_j} - f_k k \cdot \frac{\partial g}{\partial I_j} \right)^2} dI \\ &\geq \sum_{j=1}^n \sum_{k \in \mathbb{Z}^n \setminus 0} \mu \int_{\tilde{B}} \frac{1}{|ik \cdot g(I) - \mu|^2} \left| k \cdot g(I) \frac{\partial f_k}{\partial I_j} - f_k k \cdot \frac{\partial g}{\partial I_j} \right| dI. \end{aligned} \quad (36)$$

As before, we consider $\bar{I} \in \tilde{B} \cap \mathcal{R}(f)$, so that there exists $k \in \mathbb{Z}^n$ such that $k \cdot g(\bar{I}) = 0$ and $|f_k(\bar{I})| > 0$. In particular, there exist $\delta, \lambda_1, \lambda_2 > 0$ (independent of T) such that the closed ball

$$B_\delta(\bar{I}) = \{I : \|I - \bar{I}\| \leq \delta\}$$

is contained in B , and also for any $I \in B_\delta(\bar{I})$ we have

$$|f_k(I)| \geq \lambda_1,$$

and

$$\min_{\|u\|=1} \left\| \frac{\partial g^T}{\partial I} u \right\|_1 \geq \lambda_2.$$

Since $\lambda > 0$ is a Lipschitz constant for g in the set A , for any $I \in B_\delta(\bar{I})$ we also have

$$|k \cdot g(I)| \leq \|k\| \lambda \delta.$$

The series in (36) has therefore the lower bound

$$\begin{aligned}
& \mu \sum_{j=1}^n \sum_{\tilde{k} \in \mathbb{Z}^n \setminus 0} \int_{\tilde{B}} \frac{1}{|i\tilde{k} \cdot g(I) - \mu|^2} \left| \tilde{k} \cdot g(I) \frac{\partial f_{\tilde{k}}}{\partial I_j} - f_{\tilde{k}} \tilde{k} \cdot \frac{\partial g}{\partial I_j} \right| dI \\
& \geq \mu \sum_{j=1}^n \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} \left| k \cdot g(I) \frac{\partial f_k}{\partial I_j} - f_k k \cdot \frac{\partial g}{\partial I_j} \right| dI \\
& \geq \mu \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} |f_k| \left\| \frac{\partial g}{\partial I} k \right\|_1 dI - \mu \sum_{j=1}^n \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} |k \cdot g| \left| \frac{\partial f_k}{\partial I_j} \right| dI \\
& \geq \lambda_1 \lambda_2 \|k\| \mu \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} dI - \mu \|k\| \delta \lambda \left(\sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} dI \\
& = \left(\lambda_1 \lambda_2 - \delta \lambda \sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) \|k\| \mu \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} dI.
\end{aligned}$$

First, we remark that in the case $\sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty > 0$, it is not restrictive to choose δ satisfying

$$\delta \leq \frac{\lambda_1 \lambda_2}{2\lambda \sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty},$$

so that we have

$$\sum_{j=1}^n \sum_{\tilde{k} \in \mathbb{Z}^n \setminus 0} \mu \int_{\tilde{B}} \frac{1}{|i\tilde{k} \cdot g(I) - \mu|^2} \left| \tilde{k} \cdot g(I) \frac{\partial f_{\tilde{k}}}{\partial I_j} - f_{\tilde{k}} \tilde{k} \cdot \frac{\partial g}{\partial I_j} \right| dI \geq \frac{\lambda_1 \lambda_2}{2} \|k\| \mu \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} dI.$$

Then by performing the change of variables $J := g(I)$ and using (34) we obtain the lower bound

$$\begin{aligned}
& \frac{\lambda_1 \lambda_2}{2} \|k\| \mu \int_{B_\delta(\bar{I})} \frac{1}{|ik \cdot g(I) - \mu|^2} dI \geq \frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{g(B_\delta(\bar{I}))} \frac{1}{|ik \cdot J - \mu|^2} dJ \\
& = \frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{g(B_\delta(\bar{I}))} \frac{1}{\sqrt{(k \cdot J)^2 + \mu^2}} dJ
\end{aligned}$$

which, by the additional change of variables $x := RJ$ as in (29), can be written as

$$\frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{Rg(B_\delta(\bar{I}))} \frac{1}{\sqrt{\|k\|^2 x_1^2 + \mu^2}} dx.$$

Since there exists $\tilde{\delta} > 0$ possibly depending on k, \bar{I} (but independent of μ) such that

$$\left\{ x : \max_{j=1, \dots, n} |x_j - Rg(\bar{I})_j| \leq \tilde{\delta} \right\} \subseteq Rg(B_\delta(\bar{I})),$$

we obtain the lower bound

$$\frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{Rg(B_\delta(\bar{I}))} \frac{1}{\sqrt{\|k\|^2 x_1^2 + \mu^2}} dx \geq \frac{\lambda_1 \lambda_2}{2M} \|k\| \tilde{\delta}^{n-1} \mu \int_{-\tilde{\delta}}^{\tilde{\delta}} \frac{1}{\sqrt{\|k\|^2 x_1^2 + \mu^2}} dx_1$$

$$= \frac{\lambda_1 \lambda_2}{2M} \int_{-\frac{\|k\|}{\mu} \tilde{\delta}}^{\frac{\|k\|}{\mu} \tilde{\delta}} \frac{1}{1+y^2} dy = \frac{\lambda_1 \lambda_2}{M} \arctan \frac{\|k\|}{\mu} \tilde{\delta}.$$

Since we have

$$\lim_{\mu \rightarrow 0^+} \arctan \frac{\|k\|}{\mu} \tilde{\delta} = \frac{\pi}{2},$$

with a suitable definition of ϵ one immediately obtains (35).

We conclude our proof by showing the convergence of $F^{\mu, \nu}$ to \bar{f} in $A \times \mathbb{T}^n$ on sequences $(\mu_i, \nu_i) \rightarrow (0, 0)$ such that

$$\lim_{i \rightarrow 0} \frac{\mu_i}{\nu_i} = 0. \quad (37)$$

We first provide an estimate on the different contributions to

$$|F^{\mu, \nu} - \bar{f}|^1 = |F^{\mu, \nu} - \bar{f}|^0 + \sum_{k \in \mathbb{Z}^n} \sum_{j=1}^n \int_A \left| \frac{\partial}{\partial I_j} (F^{\mu, \nu} - \bar{f})_k \right| dI + \sum_{k \in \mathbb{Z}^n} \sum_{j=1}^n \int_A |k_j| |(F^{\mu, \nu} - \bar{f})_k| dI.$$

The first term $|F^{\mu, \nu} - \bar{f}|^0$ has been already estimated (see (15))

$$|F^{\mu, \nu} - \bar{f}|^0 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\|k\|} \left(1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right). \quad (38)$$

Then, for any $I \in \tilde{A}$, from (25) we have

$$(F^{\mu, \nu} - \bar{f})_k(I) = \begin{cases} F_k^{\mu, \nu}(I) = -\mu \frac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases}$$

so that we need to estimate

$$\int_{\tilde{A}} \left| \frac{\partial F_k^{\mu, \nu}}{\partial I_j}(I) \right| dI = \mu \int_{\tilde{A}} \left| \frac{\partial f_k}{\partial I_j}(I) \frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} + f_k(I) \frac{\partial}{\partial I_j} \left(\frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right) \right| dI$$

for any $k \in \mathbb{Z}^n \setminus 0$. By using the changes of variables (27) and (29) as in the proof of Proposition 2.1 –and proceeding as in estimate (31)– we obtain

$$\mu \sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \int_{\tilde{A}} \left| \frac{\partial f_k}{\partial I_j}(I) \frac{1}{ik \cdot g(I) - \mu - \nu_i \|k\|^2} \right| dI \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) \frac{\mu}{\|k\|} \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right]. \quad (39)$$

Using (1) we first obtain

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \mu \int_{\tilde{A}} \left| f_k(I) \frac{\partial}{\partial I_j} \left(\frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right) \right| dI \\ & \leq \sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \mu |f_k|^\infty \int_{\tilde{A}} \frac{1}{|ik \cdot g(I) - \mu - \nu \|k\|^2|^2} \left| \frac{\partial}{\partial I_j} (ik \cdot g(I)) \right| dI \\ & \leq \sum_{k \in \mathbb{Z}^n \setminus 0} \mu |f_k|^\infty n^2 \|k\| D \int_{\tilde{A}} \frac{1}{(k \cdot g(I))^2 + (\mu + \nu \|k\|^2)^2} dI, \end{aligned}$$

then using the change of variables (27) and (29) we have

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^n \setminus 0} \mu |f_k|^\infty n^2 \|k\| D \int_{\tilde{A}} \frac{1}{(k \cdot g(I))^2 + (\mu + \nu \|k\|^2)^2} dI \\
& \leq \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{n^2 D}{m} |f_k|^\infty \|k\| \mu \int_{g(\tilde{A})} \frac{1}{(k \cdot J)^2 + (\mu + \nu \|k\|^2)^2} dJ \\
& \leq \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{n^2 C^{n-1} D}{m} |f_k|^\infty \|k\| \mu \int_{-C}^C \frac{1}{\|k\|^2 x_1^2 + (\mu + \nu \|k\|^2)^2} dx_1 \\
& \leq \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{2n^2 C^{n-1} D}{m} |f_k|^\infty \frac{\|k\|}{(\mu + \nu \|k\|^2)^2} \mu \int_0^C \frac{1}{\frac{\|k\|^2}{(\mu + \nu \|k\|^2)^2} x_1^2 + 1} dx_1 \\
& = \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{2n^2 C^{n-1} D}{m} |f_k|^\infty \frac{\mu}{(\mu + \nu \|k\|^2)} \int_0^{\frac{\|k\| C}{\mu + \nu \|k\|^2}} \frac{1}{1 + y^2} dy \\
& = \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{2n^2 C^{n-1} D}{m} |f_k|^\infty \frac{\mu}{(\mu + \nu \|k\|^2)} \arctan \left(\frac{\|k\| C}{\mu + \nu \|k\|^2} \right).
\end{aligned}$$

From the previous inequality, we obtain

$$\sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \mu \int_{\tilde{A}} \left| f_k(I) \frac{\partial}{\partial I_j} \left(\frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right) \right| dI \leq \frac{n^2 \pi C^{n-1} D}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\mu + \nu \|k\|^2}. \quad (40)$$

In order to conclude the estimate of $|F^{\mu, \nu} - \bar{f}|^1$ it remains to consider

$$\sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \int_{\tilde{A}} \left| k_j \frac{\mu_i f_k(I)}{ik \cdot g(I) - \mu_i - \nu_i \|k\|^2} \right| dI.$$

This term is estimated by using the changes of variables (27) and (29), so that

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \mu |k_j| \int_{\tilde{A}} \left| \frac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right| dI \leq \sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{j=1}^n \frac{2\mu |k_j| |f_k|^\infty C^{n-1}}{m \|k\|} \left[l_1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right] \\
& \leq \frac{2n C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \mu \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right]. \quad (41)
\end{aligned}$$

By collecting inequalities (38), (39), (40) and (41), we obtain

$$\begin{aligned}
|F^{\mu, \nu} - \bar{f}|^1 & \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\frac{\mu}{\|k\|} \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right] |f_k|^\infty + \frac{\mu}{\|k\|} \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right] \left(\sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) \right. \\
& \quad \left. + \frac{1}{2} n^2 \pi D \frac{\mu}{\mu + \nu \|k\|^2} |f_k|^\infty + n \mu \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right] |f_k|^\infty \right) \\
& \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\mu \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right] \left((1+n) |f_k|^\infty + \sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) + \frac{1}{2} n^2 \pi D \frac{\mu}{\mu + \nu \|k\|^2} |f_k|^\infty \right)
\end{aligned}$$

so that (16) is proved. Since for $\mu, \nu > 0$ and $\|k\| \geq 1$, we have

$$\mu \log \frac{\|k\|C}{\mu + \nu \|k\|^2} \leq \mu \log \frac{C}{\nu} \leq \frac{\mu}{\nu} \left(\nu \log \frac{C}{\nu} \right)$$

and

$$\frac{\mu}{\mu + \nu \|k\|^2} \leq \frac{\mu}{\nu},$$

from (16) we obtain

$$|F^{\mu, \nu} - \bar{f}|^1 \leq \left(\frac{\mu}{\nu} \right) \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\left(\nu + \nu \log \frac{C}{\nu} \right) \left((1+n) |f_k|^\infty + \sum_{j=1}^n \left| \frac{\partial f_k}{\partial I_j} \right|^\infty \right) + \frac{1}{2} n^2 \pi D |f_k|^\infty \right).$$

Therefore, for any sequence $\mu_i, \nu_i > 0$ converging to zero with μ_i/ν_i converging to zero, we have

$$\lim_{i \rightarrow +\infty} |F^{\mu_i, \nu_i} - \bar{f}|^1 = 0.$$

The proof of Proposition 2.2 is concluded. □

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